Realizability-Aware Free Orthotropic Material Optimization: From Convex Relaxations to Hashin–Shtrikman Bounds

Marek Tyburec[1,*], **Michael Stingl**[2] **and Shenyuan Ma**[3]

[1]Faculty of Civil Engineering, Czech Technical University in Prague, Czech Republic
e-mail: marek.tyburec@cvut.cz

[2]Friedrich-Alexander University Erlangen-Nuremberg, Erlangen, Germany
e-mail: michael.stingl@fau.de

[3]Faculty of Civil Engineering, Czech Technical University in Prague, Czech Republic
e-mail: ma.shenyuan@fel.cvut.cz

KEYWORDS: *Free Material Optimization, Orthotropic Materials, Hashin–Shtrikman Bounds, Sequential Global Programming, Sequential Laminates*

Free material optimization treats the elasticity tensor field itself as the design variable. In classical compliance minimization, admissibility is usually enforced by positive semidefiniteness together with resource constraints expressed in terms of tensor invariants, typically of trace or Frobenius norm type. These constraints control the overall amount of stiffness, but they do not guarantee that an admissible tensor can be realized as the effective property of a composite made from prescribed constituent phases. This work narrows that gap for two-dimensional plane-stress problems with two well-ordered isotropic phases by introducing a hierarchy of realizability-aware admissible sets based on zeroth-order, Voigt, and Hashin–Shtrikman energy bounds.

The proposed hierarchy has several notable structural properties. In the convex setting, the Voigt admissible set is strictly tighter than the zeroth-order one for intermediate phase volume fractions, while the two coincide at the pure-phase endpoints; moreover, the Voigt model reduces to an isotropic variable-thickness-sheet formulation. For the Hashin–Shtrikman model, the energy upper bound can be written as a Voigt term minus a nonnegative correction, which explains both the strict tightening of the admissible set and the loss of joint convexity in the effective tensor and the local volume fraction. In the single-loadcase continuum setting, the resulting relaxation is tight with the classical Allaire–Kohn relaxed problem and is attained in the relaxation sense by orthotropic sequential laminates. In generic multi-loadcase settings, by contrast, it should be interpreted as a lower bound on compliance minimization over general microstructures.

Computationally, the resulting nonconvex free orthotropic material optimization problem is solved by sequential global programming. Numerical experiments illustrate the expected compliance ordering induced by the zeroth-order, Voigt, and Hashin–Shtrikman models and show that the Hashin–Shtrikman formulation remains close to finite-rank laminate reference designs while staying computationally tractable. Overall, these results connect classical homogenization bounds with free material optimization and show how realizability-aware energy bounds can inform structural design.

ACKNOWLEDGEMENT: We acknowledge financial support from the European Union through the ROBOPROX project (reg. no. CZ.02.01.01/00/22_008/0004590) and from the mobility project 8J24DE005, funded jointly by the Ministry of Education, Youth and Sports of the Czech Republic (MŠMT) and the German Academic Exchange Service (DAAD).

[1] M. Tyburec, M. Stingl, S. Ma, *Hierarchy of bounds in free orthotropic material optimization: From convex relaxations to Hashin–Shtrikman via sequential global programming*, arXiv:2602.23180, 2026.

[2] G. Allaire, R.V. Kohn, *Explicit optimal bounds on the elastic energy of a two-phase composite in two space dimensions*, *Quart. Appl. Math.* **51**(4), 675–699, 1993. https://doi.org/10.1090/qam/1247434

[3] K. Burazin, I. Crnjac, M. Vrdoljak, *Optimality criteria method in 2D linearized elasticity problems*, *Appl. Numer. Math.* **160**, 192–204, 2021. https://doi.org/10.1016/j.apnum.2020.10.002