# First- and Second-Order Strain-Gradient Materials in the Light of Singular Wave Analysis

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This paper investigates first- and second-order strain-gradient materials by means of nonlinear wave dynamics, using singular shock and acceleration waves to characterize admissible constitutive equations. The framework focuses on one-dimensional continua, where the state is described by the stress $\sigma(x,t)$, the strain $\varepsilon(x,t)$, and the velocity $v(x,t)$, subject to the momentum equation $\rho v_t = \sigma_x$, the kinematic relation $\varepsilon_t = v_x$, and a general constitutive equation

$$F(\sigma, \varepsilon, \sigma_x, \sigma_t, \varepsilon_x, \varepsilon_t) = 0$$

or

$$G(\sigma, \varepsilon, \sigma_x, \sigma_t, \sigma_{xx}, \sigma_{xt}, \sigma_{tt}, \varepsilon_{xx}, \varepsilon_{xt}, \varepsilon_{tt}) = 0$$

in the second-gradient case.

In the method of singular wave analysis, a material initially in a uniform physical state is disturbed at one point, and the resulting disturbance propagates along a wave front $\varphi(x,t) = \text{const}$ with speed $c = -\varphi_t/\varphi_x$. Using Hadamard's jump relations, the paper distinguishes between first-order (shock) waves, second-order (acceleration) waves, and third-order (higher-gradient) waves, where the jumps vanish up to a given order of derivatives but reappear at the next order. This classification allows the derivation of compatibility conditions on the wave amplitude moduli $\mu, \kappa, \nu$ and the propagation speed $c$ from the governing equations.

For second-order waves, the analysis yields a third-degree polynomial in the wave speed $c$ of the form

$$\rho \frac{\partial F}{\partial \sigma_t} c^3 - \rho \frac{\partial F}{\partial \sigma_x} c^2 + \frac{\partial F}{\partial \varepsilon_t} c - \frac{\partial F}{\partial \varepsilon_x} = 0.$$

For third-order (second-gradient) waves, the compatibility analysis is extended to second-order spatial and temporal derivatives, leading to a fourth-degree polynomial

$$2\rho \frac{\partial G}{\partial \sigma_{tt}} c^4 - \rho \frac{\partial G}{\partial \sigma_{xt}} c^3 + \left( \rho \frac{\partial G}{\partial \sigma_{xx}} + \frac{\partial G}{\partial \varepsilon_{tt}} \right) c^2 - \frac{\partial G}{\partial \varepsilon_{xt}} c + \frac{\partial G}{\partial \varepsilon_{xx}} = 0.$$

The work thus provides a systematic way to deduce admissible first- and second-order gradient constitutive equations from the structure of second- and third-order singular waves, linking the non-local nature of gradient materials to the dynamics of disturbance propagation in a unified framework.

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