Modeling metamaterials by second-order rate-type constitutive relations between only the macroscopic stress and strain

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Metamaterials are materials with specifically designed microstructure which grants them unusual properties not typically seen in nature such especially regarding wave propagation. Macroscopic behavior of such materials is typically described by homogenous linear elastic material with the effective density and stiffness depending on the frequency. The resulting effective density / stiffness can be even negative for certain frequencies. Such interpretation contradicts the principles of Newtonian mechanics that the mass is constant. A few different ways have been proposed to address this issue. These alternative models utilize concepts of "micro-level deformation", "micromorphic continua" or "enriched continua". While details may vary, the key idea is the same – introduce an additional kinematic variable.

We present yet another approach that addresses the issue, and which results in effective rate-type constitutive relations. The relations are formulated as second-order in time rate-type Eulerian constitutive relations between only the Cauchy stress tensor, the Hencky strain tensor and objective time derivatives thereof without the need to introduce additional kinematic variables. Furthermore, the proposed approach works even in finite deformations setting, and the resulting constitutive relations are thermodynamically consistent. The linearisation of the constitutive relations leads to the same Fourier representation as the model utilizing frequency dependent quantities, but with constant material parameters.

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