On physical interpretation of normalized third principal invariant of Cauchy stress deviator and its derivatives Lode parameter, skewness angle – shear stress mode angle

Andrzej Ziółkowski

Institute of Fundamental Technological Research, Polish Academy of Sciences, Warsaw, Poland
e-mail: aziolk@ippt.pan.pl

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A set of three functionally independent invariants of Cauchy stress tensor $\boldsymbol{\sigma}$, i.e. first principal invariant (I_1), second principal invariant of stress deviator (shear stress) (J_2) and normalized third invariant of stress deviator $(\bar{\mathrm{J}}_3)$ proved to be very useful in modeling behavior of solid materials,

$$\begin{aligned} &\mathrm{I}_1 \equiv 3\sigma_m = \sigma_I + \sigma_{II} + \sigma_{III}, \quad \mathrm{J}_2 \equiv \tfrac{1}{2}(s_I{}^2 + s_{II}{}^2 + s_{III}{}^2), \\ &\bar{\mathrm{J}}_3 \equiv 3\sqrt{6}\, J_3/(2\mathrm{J}_2)^{3/2} \in <-1, 1>, \quad \mathrm{J}_3 \equiv s_I s_{II} s_{III} = \tfrac{1}{3}(s_I{}^3 + s_{II}{}^3 + s_{III}{}^3), \end{aligned} \tag{1}$$

where $\sigma_K, s_K, K = I, II, III$ denote principal stresses of Cauchy stress and its deviator, respectively.

It is vital to know what actually these invariants describe regarding physical phenomena, i.e. identify their rational physical interpretation. Physical interpretation of the first principal invariant of Cauchy stress, as pressure p $(-p = \tfrac{1}{3}\mathrm{I}_1)$, is so straightforward that its identification did not require deeper thought. Comprehensive physical interpretation of second invariant of stress deviator (J_2) as a *macroscopic scalar measure* characterizing quantitatively, in an *average manner over all directions on unit sphere, shearing forces intensity* (τ_{av}), i.e. effective action of population of *micro pure shears (mechanical dipoles)* generating given *macroscopic stress state* was exposed by Novozhilov in 1951 [1]. The same physical interpretations can be assigned to derivative quantities: shear stress modulus, effective (equivalent) stress, being linearly proportional to τ_{av},

$$\begin{aligned} &\tau_{av}(\boldsymbol{\sigma}) \equiv \left((1/\Omega)\int \tau^2 d\Omega\right)^{1/2} = \sqrt{\tfrac{2}{5} J_2} \quad (\tau^2 = \sigma_I{}^2 l_I{}^2 + \sigma_{II}{}^2 l_{II}{}^2 + \sigma_{III}{}^2 l_{III}{}^2 - (\sigma_I l_I{}^2 + \sigma_{II} l_{II}{}^2 + \sigma_{III} l_{III}{}^2)^2), \\ &\tau_{\mathrm{av}} = \sqrt{\tfrac{2}{5}\mathrm{J}_2} \quad \Rightarrow \quad \|\mathbf{s}\| \equiv \sqrt{2\mathrm{J}_2} = \sqrt{5}\,\tau_{\mathrm{av}}, \quad \sigma_{\mathrm{ef}} \equiv \sqrt{3\mathrm{J}_2} = \sqrt{\tfrac{15}{2}}\,\tau_{\mathrm{av}}, \end{aligned} \tag{2}$$

where τ_{av} is the Novozhilov's *shear stress intensity*, τ is modulus of shear stress traction operating on an elementary surface of a unit sphere $d\Omega$, l_I, l_{II}, l_{III} are direction cosines determining orientation of a normal to the surface $d\Omega$ in relation to principal directions of stress tensor $\boldsymbol{\sigma}$.

The present work presents physical interpretation, currently non-existing, of the normalized third invariant of stress deviator $(\bar{\mathrm{J}}_3)$ as the *scalar macroscopic measure* describing *degree of orientational ordering (directional concentration)* in population of *micro pure shears* generating given *macroscopic stress state* around specific directions, see also Lecture 8 in [2]. The same interpretation gain functional derivates of $\bar{\mathrm{J}}_3$, i.e., Lode parameter μ, skewness angle θ_{sk} — *shear stress mode angle* defined with *pure shears as reference comparison states*, Lode angle θ_{L},

$$\theta_{sk} \equiv \tfrac{1}{3}\sin^{-1}(\bar{\mathrm{J}}_3), \;\; \mu \equiv 3s_{II}/(s_I - s_{III}) = -\sqrt{3}\,tg(\theta_{sk}), \;\; \theta_L \equiv \tfrac{1}{3}\cos^{-1}(\bar{\mathrm{J}}_3); \quad \tau_{\max} \equiv \tfrac{1}{2}(\sigma_1 - \sigma_3) = \sqrt{\tfrac{5}{2}}\,\tau_{\mathrm{av}}\cos(\theta_{\mathrm{sk}}). \tag{3}$$

[1] V.V. Novozhilov, О физическом смысле инвариантов напряжения, используемых в теории пластичности (in Russian), *Prikl. Mat. Mekh.*, XVI, 617–619, 1952.

[2] A. Ziółkowski, *On symmetric tensors of second order and fourth order in mechanics with focus on Cauchy stress and Hooke tensors*, IPPT PAN, 2025. https://wydawnictwa.ippt.pan.pl/sklep/on-symmetric-tensors-of-second-and-fourth-order-in-mechanics-/b79 (Open access). 978-83-65550-63-7.