Finite Element Modeling of Torsion in Functionally Graded Shafts

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This study investigates the Saint-Venant torsion of functionally graded (FGM) shafts using the Finite Element Method. For the purpose of this research, numerical models of shafts with circular, elliptical, equilateral triangular, and square cross-sections were developed. A functional material gradation was applied radially from the geometric center to the outer boundary, transitioning from an aluminum (Al) core to a titanium (Ti) surface. The effective material properties were evaluated using the Mori-Tanaka micromechanical model. Two distinct gradient modeling techniques were employed and compared: a conventional discrete layer-wise gradation and a novel approach utilizing a continuous field variable interpolated directly from nodes to Gauss integration points. Boundary conditions consistent with Saint-Venant free torsion theory [1] were applied, allowing for the unconstrained warping of the non-circular cross-sections. To prevent rigid body motion without overconstraining the model, a weak springs algorithm was implemented, yielding negligible reaction forces on the order of 10^{-14} N. Figure 1 shows a concept of the shaft material gradation using: a), d) the discrete layer-wise method, and b), c) the continuous method.

Figure 1. Material gradation in various cross sections using different modeling methods.

The effective material properties were evaluated using the Mori-Tanaka [2] homogenization, governed by:

$$G^{MT} = G_1 + \frac{V_f(G_2-G_1)G_1}{(1-V_f)(G_2-G_1)\beta_1+G_1}, \beta_1 = \frac{6(K_1+2G_1)}{5(3K_1+4G_1)}. \tag{1}$$

The model incorporates the reinforcement volume fraction V_f [0-1], alongside the shear and bulk moduli of Al and Ti. High-fidelity results are ensured via rigorous strain energy-based mesh convergence and GCI calculations. Subjecting the shafts to *+/-1Nm* torsional moments, the study evaluates the impact of Al/Ti gradation on cross-sectional warping and resultant shear stress distribution, using a homogeneous aluminum shaft as a baseline. Figure 2 shows results of u_z displacement for triangular cross-section.

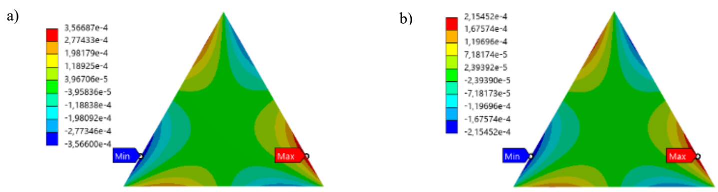

Figure 2. Results of u_z displacement for triangular cross-section, a) aluminium, b) Al/Ti FGM.

[1] De Saint-Venant, A. J. C. B. (1856). Mémoire sur la torsion des prismes. *Mémoires présentés par divers savants à l'Académie des Sciences*, 14, 233-560.

[2] Mori, Tanaka, and Kohichi Tanaka. "Average stress in matrix and average elastic energy of materials with misfitting inclusions." *Acta Metallurgica* 21.5 (1973): 571-574.