Exact Geometric Representation of Curved Domains in Discontinuous Galerkin Method

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This work presents the formulation and analysis of a Discontinuous Galerkin Method (DGM) for second-order boundary value problems defined on two-dimensional domains with curved boundaries. The method is developed within a broken-Sobolev-space framework over meshes composed of convex polygonal finite elements. In DGM, the underlying function space is discontinuous across the mesh skeleton in the infinite-dimensional setting, and this discontinuity is inherited by the corresponding finite-dimensional approximation space. Within the infinite-dimensional space, we identify the subspace consisting of continuous functions that satisfy homogeneous Dirichlet boundary conditions. In the finite setting, we determine the subspace that minimizes both the interelement discontinuities and the deviation from homogeneous Dirichlet boundary conditions [1].

In contrast to the standard FEM, the basis functions in the DGM are defined independently on each finite element, and therefore do not depend on the element's shape. Then, a wide range of basis functions can be employed, and polygonal finite elements of arbitrary geometry may be used. In particular, elements with curved edges can be incorporated to accurately represent the curved outer boundary of the computational domain. For numerical integration, each polygonal element is subdivided into triangles, to which suitable quadrature schemes are applied. Standard numerical integration rules are used for triangles with straight edges. When a triangle possesses one curved edge, integration is performed using a scaled boundary approach [2]. The outer boundary of the domain must be described analytically by a parameterized function. This parametrization is then transferred to the polygonal edges aligned with the outer boundary in order to curve those edges and to generate the corresponding numerical integration schemes for both the curved elements and the curved portions of the external boundary.

It is well known property of higher-order finite element approximations yield higher convergence rates, allowing significantly improved accuracy with comparatively fewer degrees of freedom than low-order elements. However, when the boundary of the domain is curved, small elements must often be used to faithfully recover the geometry, which diminishes the advantages of high-order elements. The approach adopted in this research enables the use of relatively large finite elements while providing an exact representation of the domain geometry. Under these conditions, the size of the finite elements is determined solely by the need to reduce the approximation error of the numerical solution, rather than by geometric constraints.

The capabilities of the method are demonstrated through a series of numerical examples. These include benchmark Poisson problems featuring highly irregular solutions on strongly curved domains. In addition, the method is applied to the Stokes problem that combines the velocity and pressure fields. The presented examples illustrate the flexibility of the approach and its numerical robustness, even for high-order approximations.

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